

ON t -EXTENSIONS OF THE HANKEL DETERMINANTS OF CERTAIN AUTOMATIC SEQUENCES

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ABSTRACT. In 1998, Allouche, Peyrière, Wen and Wen considered the Thue–Morse sequence, and proved that all the Hankel determinants of the period-doubling sequence are odd integral numbers. We speak of t -extension when the entries along the diagonal in the Hankel determinant are all multiplied by t . Then we prove that the t -extension of each Hankel determinant of the period-doubling sequence is a polynomial in t , whose leading coefficient is the *only one* to be an odd integral number. Our proof makes use of the combinatorial set-up developed by Bugeaud and Han, which appears to be very suitable for this study, as the parameter t counts the number of fixed points of a permutation. Finally, we prove that all the t -extensions of the Hankel determinants of the regular paperfolding sequence are polynomials in t of degree less than or equal to 3.

1. INTRODUCTION

Hankel determinant is a very classical mathematical subject widely studied in Linear Algebra, Combinatorics, Number Theory and Algorithmics (see, for example, [12, 17, 13, 8, 7]). In particular, the Hankel determinants of automatic sequences play an important role in the study of irrationality exponents in Number Theory. The first result on such determinants, obtained in 1998, is due to Allouche, Peyrière, Wen and Wen [2], who considered the Thue–Morse sequence, and proved that all the Hankel determinants of the period-doubling sequence are odd integral numbers. This result allowed Bugeaud [3] to prove that the irrationality exponents of the Thue–Morse–Mahler numbers are exactly 2.

Let x be a parameter. We identify each sequence $\mathbf{c} = (c_0, c_1, c_2, \dots)$ with its generating function $C = C(x) = c_0 + c_1x + c_2x^2 + \dots$. In general, the constant term c_0 will be equal to 1. For $k \geq 1$ and $p \geq 0$

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let

$$(1) \quad H_k^p(C) = H_k^p(\mathbf{c}) := \begin{vmatrix} c_p & c_{p+1} & \cdots & c_{p+k-1} \\ c_{p+1} & c_{p+2} & \cdots & c_{p+k} \\ \vdots & \vdots & \ddots & \vdots \\ c_{p+k-1} & c_{p+k} & \cdots & c_{p+2k-2} \end{vmatrix}$$

be the (p, k) -order Hankel determinant of the series $C(x)$ or of the sequence $\mathbf{c} = (c_0, c_1, c_2, \dots)$. We write $H_k(C) := H_k^0(C)$ for short. The Thue–Morse sequence $\mathbf{e} = (1, -1, -1, 1, \dots)$ can be defined by the generating function

$$(2) \quad P_2(x) = \sum_{k=0}^{\infty} e_k x^k = \prod_{k=0}^{\infty} (1 - x^{2^k}).$$

Then, the *period-doubling sequence* $\mathbf{d} = (1, 0, 1, 1, 1, 0, \dots)$ is derived from the Thue–Morse sequence by defining

$$(3) \quad d_k = \frac{1}{2} |e_k - e_{k+1}| \quad (k \geq 0).$$

Theorem 1 (APWW). *For every positive integer k the Hankel determinant $H_k(\mathbf{d})$ of the period-doubling sequence \mathbf{d} is an odd integral number. In other words,*

$$(4) \quad H_k(\mathbf{d}) \equiv 1 \pmod{2}.$$

Coons [5] considered the series

$$(5) \quad G_{0,0}(x) := \sum_{n=0}^{\infty} \frac{x^{2^n-1}}{1 - x^{2^n}}$$

and proved that all the Hankel determinants $H_k(G_{0,0})$ of the power series $G_{0,0}(x)$ are odd integral numbers. As shown in [4], Coons’s result is essentially equivalent to Theorem 1.

Let t be a parameter. We speak of *t -extension* when the entries along the diagonal in the (p, k) -order Hankel determinant are all multiplied by t . In other words, we define the *t -Hankel determinant* of the formal power series $C(x) = c_0 + c_1x + c_2x^2 + \cdots$ by

$$(6) \quad H_k^p(C, t) := \begin{vmatrix} c_p t & c_{p+1} & \cdots & c_{p+k-1} \\ c_{p+1} & c_{p+2} t & \cdots & c_{p+k} \\ \vdots & \vdots & \ddots & \vdots \\ c_{p+k-1} & c_{p+k} & \cdots & c_{p+2k-2} t \end{vmatrix}.$$

Obviously, the above t -Hankel determinant (6) is a polynomial in t of degree less than or equal to k , which is equal to the traditional Hankel determinant (1) when $t = 1$. Again, we write $H_k(C, t) := H_k^0(C, t)$. Our main result is stated as follows.

Theorem 2. *For every positive integer k the t -Hankel determinant $H_k(\mathbf{d}, t)$ of the period-doubling sequence \mathbf{d} is a polynomial in t of degree k , whose leading coefficient is the only one to be an odd integral number. In other words,*

$$(7) \quad H_k(\mathbf{d}, t) \equiv t^k \pmod{2}.$$

In the following table we reproduce the first values of the t -Hankel determinants of the period-doubling sequence \mathbf{d} . We see that all the coefficients are even integral numbers, except the coefficient of t^k . When $t = 1$ we recover Theorem 1.

k	$H_k(\mathbf{d}, t)$	$H_k(\mathbf{d}, t) \pmod{2}$	$H_k(\mathbf{d}, 1)$
0	1	1	1
1	t	t	1
2	t^2	t^2	1
3	$t^3 - 2t$	t^3	-1
4	$t^4 - 4t^2$	t^4	-3
5	$t^5 - 6t^3 + 2t^2 + 4t$	t^5	1
6	$t^6 - 8t^4 + 4t^3 + 12t^2 - 8t$	t^6	1
7	$t^7 - 12t^5 + 10t^4 + 24t^3 - 24t^2$	t^7	-1
8	$t^8 - 16t^6 + 16t^5 + 48t^4 - 64t^3$	t^8	-15

Actually, Theorem 1 has three proofs. The first one is due to Al-louche, Peyrière, Wen and Wen by using determinant manipulation [2], which consists of proving sixteen recurrence relations between determinants. The second one is a combinatorial proof derived by Bugeaud and Han [4]. The third proof is very short by using the Jacobi continued fraction algebra [10]. For proving Theorem 2 it seems the method used in the second proof is more suitable, as the parameter t counts the number of fixed points of permutations.

Some basic notations and properties on permutations and involutions are collected in Section 2, including the statement of the key combinatorial result, namely, Theorem 5. The proof of the main result (Theorem 2) is found in Section 3, after proving Theorem 5.

The *regular paperfolding sequence* $\mathbf{r} = (1, 1, 0, 1, 1, 0, 0, \dots)$ can be defined by the generating function [16, 1]

$$(8) \quad G_{0,2}(x) = \sum_{n \geq 0} r_n x^n = \sum_{n=0}^{\infty} \frac{x^{2^n-1}}{1 - x^{2^{n+2}}}.$$

Coons and Vrbik conjectured [6] and Guo, Wu and Wen [9] proved the following result.

Theorem 3 (GWW). *The Hankel determinants of the regular paperfolding sequence \mathbf{r} are periodic of period 10. More precisely, we have*

$$(9) \quad (H_k(\mathbf{r}))_{k=0,1,\dots} \equiv (1, 1, 1, 0, 0, 1, 0, 0, 1, 1)^* \pmod{2}.$$

An automatic proof of Theorem 3 by a computer algebra system is described in [11]. Our second result is stated next.

Theorem 4. *For every positive integer k the t -Hankel determinant $H_k(\mathbf{r}, t)$ of the regular paperfolding sequence \mathbf{r} is a polynomial in t of degree less than or equal to 3.*

Theorem 4 is proved in Section 4. In the following table we reproduce the first values of the t -Hankel determinants of the regular paperfolding sequence \mathbf{r} . We see that all the $H_k(\mathbf{r}, t)$'s are polynomials of degree less than or equal to 3.

k	$H_k(\mathbf{r}, t)$	k	$H_k(\mathbf{r}, t)$
0	1	5	$-t^3 + 2t^2 + 2t - 2$
1	t	6	$2t^2 - 2t - 4$
2	-1	7	$3t^3 - 6t^2 - 7t + 6$
3	$-2t$	8	$-9t^2 + 12t + 16$
4	$-t^2 + 2t + 1$	9	$-15t^3 + 20t^2 + 46t - 40$

As earlier mentioned, Theorem 2 is a t -extension of Theorem 1. However, Theorem 3 can not be obtained from Theorem 4 by specializing $t = 1$. The following problem remains unsolved.

Problem. *Find a true t -extension of Theorem 3. In other words, find a property of the t -Hankel determinants of the regular paperfolding sequence, which implies relation (9) when $t = 1$.*

2. PERMUTATIONS AND INVOLUTIONS

A combinatorial set-up, based on permutations and involutions, for studying the Hankel determinants of the period-doubling sequence was introduced in [4]. We propose a refinement of such a combinatorial set-up for studying t -Hankel determinants. The following infinite sets of integers play an important role.

$$N = \mathbb{N}^0 = \{0, 1, 2, 3, \dots\},$$

$$J = \{(2n+1)2^{2k} - 1 \mid n, k \in N\} = \{0, 2, 3, 4, 6, 8, 10, 11, 12, 14, \dots\},$$

$$J^* = \{(2n+1)2^{2k} - 1 \mid n, k \in N, k > 0\} = \{3, 11, 15, 19, 27, 35, \dots\},$$

$$K = N \setminus J = \{(2n+1)2^{2k+1} - 1 \mid n, k \in N\} = \{1, 5, 7, 9, 13, 17, \dots\},$$

$$L = N \setminus J^* = K \cup \{2n \mid n \in N\} = \{0, 1, 2, 4, 6, 8, 10, 13, 14, 16, \dots\},$$

$$P = \{k \mid k \equiv 0, 3 \pmod{4}\} = \{0, 3, 4, 7, 8, 11, 12, 15, 16, \dots\},$$

$$Q = \{k \mid k \equiv 1, 2 \pmod{4}\} = \{1, 2, 5, 6, 9, 10, 13, 14, 17, \dots\}.$$

For each infinite set A let $A|_m$ be the finite set composed of the smallest m integers in A .

Let $\mathfrak{S}_m = \mathfrak{S}_{\{0,1,\dots,m-1\}}$ be the set of all permutations on $N|_m$. A permutation is represented by the product of its disjoint cycles.

For example, the permutation $\sigma = (0, 5)(1)(2, 6, 3)(4, 8)(7)$ is an element from \mathfrak{S}_9 . An *involution* is a permutation σ such that $\sigma = \sigma^{-1}$. Equivalently, a permutation σ is an involution if each cycle of σ is either a fixed point (b) or a *transposition* (c, d) . For instance, $\sigma = (0, 5)(1)(2, 6)(3)(4, 8)(7) \in \mathfrak{S}_9$ is an involution. For each set B , a transposition (c, d) is said “in B ” if $c + d \in B$. In this case, we write $(c, d) \in B$.

For a nonnegative integer k and two sets of positive integers A, B such that A is finite, let $\mu(A, k, B)$ be the number of involutions σ in \mathfrak{S}_A having exactly k transpositions such that all transpositions of σ are in B . The following key result is useful for proving Theorem 2 (see Section 3).

Theorem 5. *For $m \geq 1$ and $k \geq 0$, we have*

$$(10) \quad \mu(N|_m, k, J) \equiv \begin{cases} 1 & (\text{mod } 2), \quad \text{if } k = 0; \\ 0 & (\text{mod } 2), \quad \text{if } k \geq 1. \end{cases}$$

The proof of Theorem 5 is given in Section 3, with the help of several lemmas stated in the remainder of this section.

Lemma 6. *For $m \geq 1$ and $k \geq 0$ we have*

$$(11) \quad \mu(N|_m, k, J) = \mu(P|_m, k, L)$$

and

$$(12) \quad \mu(P|_m, k, J^*) = \mu(Q|_m, k, J^*).$$

Proof. We define two transformations:

$$\begin{aligned} \beta : N \rightarrow P; \quad \ell &\mapsto \begin{cases} 2\ell, & \text{if } \ell \text{ is even;} \\ 2\ell + 1, & \text{if } \ell \text{ is odd;} \end{cases} \\ \delta : P \rightarrow Q; \quad \ell &\mapsto \begin{cases} \ell + 1, & \text{if } \ell \text{ is even;} \\ \ell - 1, & \text{if } \ell \text{ is odd.} \end{cases} \end{aligned}$$

The transformation β is a bijection of $N|_m$ onto $P|_m$, and can be extended to the set of all involutions on $N|_m$ by applying β on every letter of the involutions. For example

$$\beta((7)(0, 5), (6, 3), (1), (8, 2), (4)) = (15)(0, 11)(12, 7)(3)(16, 4)(8).$$

We now claim that, for any $c, d \in N|_m$, the transposition (c, d) is in J if and only if $(\beta(c), \beta(d))$ is in L . The proof of this claim works by distinguishing the parities of c and d : (i) if c and d are even, then $\beta(c) = 2c$ and $\beta(d) = 2d$, so that $\beta(c) + \beta(d)$ is even and is in L ; (ii) if c and d are odd, then $\beta(c) = 2c + 1$ and $\beta(d) = 2d + 1$, so that $\beta(c) + \beta(d)$ is even and is in L ; (iii) if $c + d \in J$ and one of the integers c, d is even, the other being odd. Then,

$$\beta(c) + \beta(d) = 2c + 2d + 1 = 2 \times ((2n+1)2^{2k} - 1) + 1 = (2n+1)2^{2k+1} - 1 \in L.$$

The “reverse part” is proved in the same manner. Thus, equation (11) holds.

The transformation δ is a bijection of $P|_m$ onto $Q|_m$, and can be extended to the set of all involutions on $P|_m$ by applying δ on every letter of the involutions. For example

$$\delta((15)(0, 11)(12, 7)(4)(16, 3)(8)) = (14)(1, 10)(13, 6)(5)(17, 2)(9).$$

If the transposition (c, d) is in J^* and $c, d \in P$, then one of the integers c, d is even, the other being odd. Hence,

$$\delta(c) + \delta(d) = c - 1 + d + 1 = c + d \in J^*.$$

Thus, equation (12) is proved. \square

Lemma 7. *For each $k \geq 0$ we have*

$$(13) \quad \mu(N|_{2n}, k, J^*) \equiv \begin{cases} 0 \pmod{2}, & \text{if } k \text{ is odd;} \\ \mu(P|_n, k/2, J^*) \pmod{2}, & \text{if } k \text{ is even.} \end{cases}$$

Proof. It is easy to see that, if $c + d \in J^*$, then $c + d \equiv 3 \pmod{4}$. Thus, both c and d belong either to P or to Q . Hence,

$$(14) \quad \mu(N|_{2n}, k, J^*) = \sum_{i+j=k} \mu(P|_n, i, J^*) \mu(Q|_n, j, J^*)$$

$$(15) \quad = \sum_{i+j=k} \mu(P|_n, i, J^*) \mu(P|_n, j, J^*).$$

The last identity holds by Lemma (6). When $k = 2\ell + 1$ is odd, the right-hand side of equation (15) is equal to

$$2 \sum_{i=0}^{\ell} \mu(P|_n, i, J^*) \mu(P|_n, 2\ell + 1 - i, J^*) \equiv 0 \pmod{2}.$$

When $k = 2\ell$ is even, we have

$$\begin{aligned} & \sum_{i+j=2\ell} \mu(P|_n, i, J^*) \mu(P|_n, j, J^*) \\ &= 2 \sum_{i=0}^{\ell-1} \mu(P|_n, i, J^*) \mu(P|_n, 2\ell - i, J^*) + \mu(P|_n, \ell, J^*) \mu(P|_n, \ell, J^*) \\ &\equiv \mu(P|_n, k/2, J^*) \pmod{2}. \end{aligned}$$

This achieves the proof. \square

In the sequel, the notation $a \equiv b$ means that the integers a and b are congruent modulo 2 when nothing else is specified.

Lemma 8. *For $m \geq 1$ and $k \geq 1$ we have*

$$(16) \quad \sum_{i=0}^k \mu(P|_m, i, J^*) \binom{m-2i}{2k-2i} \equiv \mu(P|_m, k, L) \pmod{2}.$$

Proof. Recall that $\mu(A, k, B)$ is the number of involutions σ in \mathfrak{S}_A having exactly k transpositions such that all transpositions of σ are in B . For two disjoint sets of integers B_1 and B_2 , we define $\mu(A, k_1, k_2, B_1, B_2)$ to be the number of involutions σ in \mathfrak{S}_A having exactly k_1 transpositions in B_1 and k_2 transpositions in B_2 such that all transpositions are in $B_1 \cup B_2$. So that $\mu(A, 0, k_2, B_1, B_2) = \mu(A, k_2, B_2)$.

Let i and j be two positive integers such that $0 \leq i \leq j \leq k$. Consider the set \mathfrak{I}_j of involutions σ on $P|_m$ having exactly j transpositions in J^* and $k - j$ transpositions in L and no other transposition. Then, the cardinality of \mathfrak{I}_j is equal to $\mu(P|_m, j, k - j, J^*, L)$. A marked involution is obtained from an involution $\sigma \in \mathfrak{I}_j$ by coloring i transpositions among the j transpositions in J^* . Let $\mathfrak{I}_{i,j}$ be the set of all those marked involutions. The cardinality of $\mathfrak{I}_{i,j}$ is equal to $\binom{j}{i} \mu(P|_m, j, k - j, J^*, L)$. Hence, the cardinality of the set $\mathfrak{I}_{i,\bullet} = \mathfrak{I}_{i,i} + \mathfrak{I}_{i,i+1} + \cdots + \mathfrak{I}_{i,k}$ is equal to

$$(17) \quad \sum_{j=i}^k \binom{j}{i} \mu(P|_m, j, k - j, J^*, L).$$

On the other hand, the marked involutions in $\mathfrak{I}_{i,\bullet}$ can be enumerated as follows. Consider the involutions on $P|_m$ that have exactly i transpositions in J^* , which are said to be colored. There are $\mu(P|_m, i, J^*)$ such involutions. Then randomly choose $2k - 2i$ letters from the rest $m - 2i$ original fixed points on $P|_m$, to generate another $k - i$ transpositions, which are either in J^* or in L . We get a marked involution which has exactly $i + (k - i) = k$ transpositions. Hence, the cardinality of the set $\mathfrak{I}_{i,\bullet}$ is equal to

$$(18) \quad \mu(P|_m, i, J^*) \binom{m - 2i}{2k - 2i} (2k - 2i - 1)(2k - 2i - 3) \cdots 3 \cdot 1.$$

Hence, the two quantities (17) and (18) are equal. We have successively

$$\begin{aligned} & \sum_{i=0}^k \mu(P|_m, i, J^*) \binom{m - 2i}{2k - 2i} \\ & \equiv \sum_{i=0}^k \mu(P|_m, i, J^*) \left[\binom{m - 2i}{2k - 2i} (2k - 2i - 1)(2k - 2i - 3) \cdots (3)(1) \right] \\ & = \sum_{i=0}^k \sum_{j=i}^k \binom{j}{i} \mu(P|_m, j, k - j, J^*, L) \\ & = \sum_{j=0}^k \left(\sum_{i=0}^j \binom{j}{i} \right) \mu(P|_m, j, k - j, J^*, L) \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=0}^k 2^j \mu(P|_m, j, k-j, J^*, L) \\
&\equiv \mu(P|_m, 0, k, J^*, L) \\
&= \mu(P|_m, k, L).
\end{aligned}$$

This achieves the proof. \square

3. PROOFS OF THEOREMS 5 AND 2

Firstly, we establish two lemmas about congruences for binomial coefficients.

Lemma 9. *For $n, k \geq 0$ we have*

$$(19) \quad \sum_{i+j=k} \binom{n}{2i} \binom{n}{2j} \equiv \begin{cases} 0 \pmod{2}, & \text{if } k \text{ is odd;} \\ \binom{n}{k} \pmod{2}, & \text{if } k \text{ is even.} \end{cases}$$

Proof. If $k = 2\ell + 1$ is odd, then

$$\sum_{i+j=2\ell+1} \binom{n}{2i} \binom{n}{2j} = 2 \sum_{i=0}^{\ell} \binom{n}{2i} \binom{n}{4\ell+2-2i} \equiv 0 \pmod{2}.$$

If $k = 2\ell$ is even, then

$$\begin{aligned}
\sum_{i+j=2\ell} \binom{n}{2i} \binom{n}{2j} &= 2 \sum_{i=0}^{\ell-1} \binom{n}{2i} \binom{n}{4\ell-2i} + \binom{n}{2\ell} \binom{n}{2\ell} \\
&\equiv \binom{n}{k} \pmod{2}.
\end{aligned}$$

This achieves the proof. \square

Lemma 10. *For $n, m, k \geq 0$ such that $n + m$ is odd we have*

$$(20) \quad \sum_{i+j=k} \binom{n}{2i} \binom{m}{2j} \equiv \binom{n+m}{2k} \pmod{2}.$$

Proof. We have

$$\begin{aligned}
\binom{n+m}{2k} &= \sum_{i+j=2k} \binom{n}{i} \binom{m}{j} \\
&= \sum_{i+j=k} \binom{n}{2i} \binom{m}{2j} + \sum_{i+j=k-1} \binom{n}{2i+1} \binom{m}{2j+1}.
\end{aligned}$$

Since $\binom{2a}{2b+1}$ is even for any positive integers a and b [14],

$$(21) \quad \binom{n}{2i+1} \binom{m}{2j+1} \equiv 0 \pmod{2},$$

if n or m is even. This is true because $n + m$ is odd. Equation (20) holds. \square

Secondly, we prove Theorem 5 by induction.

Proof of Theorem 5. When $k = 0$, the quantity $\mu(N|_m, k, J)$ counts the involutions σ without any transposition. It means that every letter of σ is a fixed point, so that $\mu(N|_m, 0, J) = 1$.

When $k \geq 1$, two cases are to be considered. Notice that any transposition of type $(\text{even}, \text{even})$ or (odd, odd) is in J since J contains all even integers. Let $k_1 + k_2 = k$. An involution σ having exactly k transpositions in J can be generated from an involution τ having exactly k_1 transpositions in J^* by adding k_2 transpositions in $J \setminus J^*$. The latter k_2 transpositions are of type $(\text{even}, \text{even})$ or (odd, odd) , and are easy to count by using binomial coefficients.

(i) When $m = 2n$ is even and $k \geq 1$ we have

$$\begin{aligned} & \mu(N|_{2n}, k, J) \\ &= \sum_{k_1+k_2=k} \mu(N|_{2n}, k_1, J^*) \sum_{i+j=k_2} \left[\binom{n-k_1}{2i} (2i-1)(2i-3) \cdots 1 \right. \\ & \quad \left. \times \binom{n-k_1}{2j} (2j-1)(2j-3) \cdots 1 \right] \\ &\equiv \sum_{k_1+k_2=k} \mu(N|_{2n}, k_1, J^*) \sum_{i+j=k_2} \binom{n-k_1}{2i} \binom{n-k_1}{2j} \pmod{2}. \end{aligned}$$

If k is odd, then one of the k_1, k_2 is odd and the other is even. By Lemma 9 and Lemma 7, $\mu(N|_{2n}, k, J) \equiv 0 \pmod{2}$. If $k = 2\ell$ is even, then

$$\begin{aligned} & \mu(N|_{2n}, k, J) \\ &= \sum_{k_1+k_2=2\ell} \mu(N|_{2n}, k_1, J^*) \sum_{i+j=k_2} \binom{n-k_1}{2i} \binom{n-k_1}{2j} \\ &\equiv \sum_{k_1+k_2=\ell} \mu(N|_{2n}, 2k_1, J^*) \sum_{i+j=2k_2} \binom{n-2k_1}{2i} \binom{n-2k_1}{2j} \text{ [By Lemma 9]} \\ &\equiv \sum_{k_1+k_2=\ell} \mu(P|_n, k_1, J^*) \binom{n-2k_1}{2k_2} \text{ [By Lemmas 7 and 9]} \\ &\equiv \mu(P|_n, \ell, L) \text{ [By Lemma 8]} \\ &= \mu(N|_n, k/2, J) \text{ [By Lemma 6]} \\ &\equiv 0 \pmod{2}. \text{ [By induction]} \end{aligned}$$

(ii) When $m = 2n + 1$ is odd and $k \geq 1$, we successivly have

$$\begin{aligned} & \mu(N|_{2n+1}, k, J) \\ &= \sum_{k_1+k_2=k} \mu(N|_{2n+1}, k_1, J^*) \sum_{i+j=k_2} \left[\binom{n+1-k_1}{2i} (2i-1)(2i-3) \cdots 1 \right. \end{aligned}$$

$$\begin{aligned}
& \times \binom{n-k_1}{2j} (2j-1)(2j-3)\cdots 1 \Big] \\
& \equiv \sum_{k_1+k_2=k} \mu(N|_{2n+1}, k_1, J^*) \sum_{i+j=k_2} \binom{n+1-k_1}{2i} \binom{n-k_1}{2j} \\
& \equiv \sum_{k_1+k_2=k} \left[\sum_{r+s=k_1} \mu(P|_{n+1}, r, J^*) \mu(Q|_n, s, J^*) \right] \binom{2n+1-2k_1}{2k_2},
\end{aligned}$$

where the last identity is obtained by using Lemma 10. Applying Lemmas 6 and 10 to the above quantity we get

$$\begin{aligned}
& \mu(N|_{2n+1}, k, J) \\
& \equiv \sum_{k_1+k_2=k} \left[\sum_{r+s=k_1} \mu(P|_{n+1}, r, J^*) \mu(P|_n, s, J^*) \right] \\
& \quad \times \sum_{i+j=k_2} \binom{n+1-2r}{2i} \binom{n-2s}{2j} \\
& = \sum_{r+s+i+j=k} \mu(P|_{n+1}, r, J^*) \binom{n+1-2r}{2i} \mu(P|_n, s, J^*) \binom{n-2s}{2j} \\
& = \sum_{k_1+k_2=k} \left[\sum_{r+i=k_1} \mu(P|_{n+1}, r, J^*) \binom{n+1-2r}{2i} \right. \\
& \quad \left. \times \sum_{s+j=k_2} \mu(P|_n, s, J^*) \binom{n-2s}{2j} \right] \\
& \equiv \sum_{k_1+k_2=k} \mu(P|_{n+1}, k_1, L) \mu(P|_n, k_2, L) \quad [\text{By Lemma 8}] \\
& \equiv \sum_{k_1+k_2=k} \mu(N|_{n+1}, k_1, J) \mu(N|_n, k_2, J). \quad [\text{By Lemma 6}] \\
& \equiv 0 \pmod{2}. \quad [\text{By induction}]
\end{aligned}$$

This achieves the proof. \square

Lastly, Theorem 2 on the t -extensions of the Hankel determinants of the period-doubling sequence is proved as follows. Keep in mind the infinite set

$$J = \{(2n+1)2^{2k} - 1 | n, k \in \mathbb{N}\} = \{0, 2, 3, 4, 6, 8, 10, 11, 12, 14, \dots\},$$

and the period-doubling sequence defined by (3). In [4] Bugeaud and Han proved the following result.

Lemma 11. *For $k \geq 0$, the integer d_k is odd if, and only if, k is in J .*

Proof of Theorem 2. Let $D(x)$ be the generating function of the period-doubling sequence

$$D(x) = \sum_{k \geq 0} d_k x^k = 1 + x^2 + x^3 + x^4 + x^6 + \cdots$$

Let k be a positive integer. By Leibniz formula for determinants [15], the t -Hankel determinant $H_k(D, t)$ is equal to

$$(22) \quad \sum_{\sigma \in \mathfrak{S}_k} t^{\text{fix}(\sigma)} (-1)^{\text{inv}(\sigma)} j_{0+\sigma(0)} j_{1+\sigma(1)} \cdots j_{k-1+\sigma(k-1)},$$

where $\text{inv}(\sigma)$ is the number of inversions of σ and $\text{fix}(\sigma)$ is the number of fixed points of σ defined by

$$\begin{aligned} \text{inv}(\sigma) &= \#\{(i, j) \mid 0 \leq i < j \leq k-1, \sigma(i) > \sigma(j)\}; \\ \text{fix}(\sigma) &= \#\{i \mid 0 \leq i \leq k-1, \sigma(i) = i\}. \end{aligned}$$

The product

$$j_{0+\sigma(0)} j_{1+\sigma(1)} \cdots j_{k-1+\sigma(k-1)}$$

is equal to 1 if $i + \sigma(i) \in J$ for $i = 0, 1, \dots, k-1$, and is equal to 0 otherwise. Let σ be a permutation such that $\sigma \neq \sigma^{-1}$. We have $\text{inv}(\sigma) = \text{inv}(\sigma^{-1})$ and $\text{fix}(\sigma) = \text{fix}(\sigma^{-1})$. Accordingly, they have the same contribution to summation (22), and can be deleted. Hence

$$(23) \quad H_k(D, t) \equiv \sum_{\sigma} t^{\text{fix}(\sigma)} j_{0+\sigma(0)} j_{1+\sigma(1)} \cdots j_{k-1+\sigma(k-1)} \pmod{2},$$

where the sum is over the set of all involutions σ on $N|_k$. By Theorem 5,

$$H_k(D, t) \equiv \sum_{i=0}^k t^{k-2i} \mu(N|_k, i, J) \equiv t^k \mu(N|_k, 0, J) = t^k.$$

This achieves the proof. \square

4. REGULAR PAPERFOLDING SEQUENCE

We define the infinite set

$$R = \{(4k+1)2^n - 1 \mid n, k \in \mathbb{N}\} = \{0, 1, 3, 4, 7, 8, 9, 12, 15, 16, \dots\}.$$

Notice that, for each integer m in the set R , there are unique integers n and k such that $(4k+1)2^n - 1 = m$. Recall the regular paperfolding sequence $\mathbf{r} = (r_k)_{k=0,1,2,\dots}$ defined by (8). We have the following result.

Lemma 12. *For each $k \geq 0$ the integer r_k is equal to 1 if and only if k is in R , and is equal to 0 otherwise.*

Proof. By definition of (8), we have

$$G_{0,2}(x) = \sum_{n \geq 0} r_n x^n = \sum_{n=0}^{\infty} \frac{x^{2^n-1}}{1 - x^{2^{n+2}}}$$

$$\begin{aligned}
&= \sum_{n=0}^{\infty} x^{2^n-1} \left(\sum_{k \geq 0} (x^{2^{n+2}})^k \right) \\
&= \sum_{n,k \geq 0} x^{4k \cdot 2^n + 2^n - 1}.
\end{aligned}$$

Thus the lemma holds. \square

Proof of Theorem 4. As discussed in Section 3, the t -Hankel determinant $H_k(\mathbf{r}, t)$ is equal to

$$(24) \quad \sum_{\sigma \in \mathfrak{S}_k} t^{\text{fix}(\sigma)} (-1)^{\text{inv}(\sigma)} r_{0+\sigma(0)} r_{1+\sigma(1)} \cdots r_{k-1+\sigma(k-1)}.$$

The product

$$r_{0+\sigma(0)} r_{1+\sigma(1)} \cdots r_{k-1+\sigma(k-1)}$$

is equal to 1 if $i + \sigma(i) \in R$ for $i = 0, 1, \dots, k-1$, and is equal to 0 otherwise.

Recall the three representations for permutations: the *one-line*, *two-line* and *product of disjoint cycles*. For example, we write

$$\sigma \in \mathfrak{S}_9 = 516280374 = \begin{pmatrix} 012345678 \\ 516280374 \end{pmatrix} = (0, 5)(1)(2, 6, 3)(4, 8)(7).$$

Consider a permutation σ having at least 4 fixed points, i.e., $\text{fix}(\sigma) \geq 4$. It's easy to know that an even number m is in R if and only if $m \equiv 0 \pmod{4}$, so that all fixed points are even. Consequently, there are at least 3 columns of type $\begin{pmatrix} \text{odd} \\ \text{odd} \end{pmatrix}$ in the two-line representation of the permutation σ . Let $\begin{pmatrix} i_1 \\ j_1 \end{pmatrix}$, $\begin{pmatrix} i_2 \\ j_2 \end{pmatrix}$ and $\begin{pmatrix} i_3 \\ j_3 \end{pmatrix}$ be the first three such columns. By the Pigeonhole Principle, there are at least two numbers among j_1, j_2, j_3 which are congruent modulo 4. Without loss of generality, we assume that j_1 and j_2 are congruent modulo 4. (When all three numbers are congruent, we also choose j_1 and j_2). We define another permutation τ obtained from σ by exchanging j_1 and j_2 in the bottom line. This procedure is reversible. And we have $\text{fix}(\sigma) = \text{fix}(\tau)$ and $\text{inv}(\sigma) + \text{inv}(\tau) \equiv 1 \pmod{2}$. Then $(-1)^{\text{inv}(\sigma)} + (-1)^{\text{inv}(\tau)} = 0$. Thus, we can delete the pair $\{\sigma, \tau\}$ from the symmetry group \mathfrak{S}_k . The value of the t -Hankel determinant $H_k(\mathbf{r}, \mathbf{t})$ defined by (24) doesn't change.

After deleting all the permutations such that $\text{fix}(\sigma) \geq 4$, all remaining permutations have at most 3 fixed points. Thus, the t -Hankel determinant $H_k(\mathbf{r}, \mathbf{t})$ is a polynomial in t of degree less than or equal to 3. \square

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